*On the Definition of Multiplication of Irrational Numbers.**

By Oswald Veblen.

In developing the theory of the irrational real numbers it is usual to start with the set of all rational numbers and to let each cut in this system define an irrational number. A cut (x, y) may be defined as a pair of sets of rational numbers $[x]$ and $[y]$ such that every rational number is either an x or a y and such that $x < y$ for every x and every y. The order relations of the irrational numbers are determined by the definition: $(x_1, y_1) < (x_2, y_2)$ if and only if there exist an x_2 and a y_1 such that $y_1 < x_2$. The sum of the two numbers defined by cuts (x_1, y_1) and (x_2, y_2) is defined as the number associated with the cut $(x_1 + x_2,\ y_1 + y_2)$. The product of two positive irrationals, (x_1, y_1) and (x_2, y_2), is defined as the number associated with the cut $(x_1 x_2',\ y_1 y_2)$, where $[x_2']$ is the set of all positive numbers in $[x_2]$; the product of two negative irrationals and of a positive and a negative irrational is defined similarly.

These definitions of addition and multiplication are stated in the form of arbitrary rules. They are, however, designed so that all the algebraic laws of operation (associative, distributive, commutative, etc.)† shall be satisfied, so that whenever two rational numbers are added or multiplied the result is the ordinary sum or product of the two rationals, and also so that when a rational number a is added or multiplied to an irrational (x, y) the results are the irrational numbers defined by the cuts $(a + x,\ a + y)$ and $(a x,\ a y)$. The question arises: *Is it possible to define the addition and multiplication of irrationals so as to satisfy these requirements, and yet so as to give a different result from the ordinary one when irrationals are combined with irrationals?*

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† Of course, if we were to require that addition and multiplication shall preserve order relations (*e. g.*, $ab > 0$, whenever $a > 0$, $b > 0$, etc.), it is obvious that the ordinary definitions would result. But the conditions we are inquiring about merely require that order relations shall be preserved when rational numbers are combined with rationals or irrationals, and our question is whether it is possible to infer anything about the case in which irrationals are combined with irrationals.

The answer is that it is possible. In fact, we can impose the further requirement that addition of irrationals shall proceed by the rule

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

This result depends on the theorem* that there exists a well-ordered sequence of irrational numbers

$$a_1, a_2, \ldots, a_\omega, a_{\omega+1}, \ldots, \tag{1}$$

such that every real number is expressible uniquely in the form

$$\alpha_0 + \alpha_1 a_{i_1} + \alpha_2 a_{i_2} + \ldots + \alpha_n a_{i_n}, \tag{2}$$

where there are a finite number of terms and the α's are rational.

The ordinary rule for multiplying irrationals described in the first paragraph determines a multiplication table for the a's, that is to say, a set of rules of the form

$$a_i a_j = \beta_0 + \beta_1 a_{i_1} + \beta_2 a_{i_2} + \ldots + \beta_m a_{i_m}. \tag{3}$$

The product of any two irrationals is obtained by expressing them in the form (2), multiplying according to the rule for multiplying polynomials and reducing the result according to the multiplication table.

Now suppose we denote by

$$a'_1, a'_2, a'_3, \ldots, a'_\omega, a'_{\omega+1}, \ldots \tag{4}$$

the a's arranged in another order of the same type as (1). Such an order would be obtained, for example, by interchanging a_1 and a_2 and leaving the other a's unaltered. Every real number is expressible uniquely in the form

$$\alpha_0 + \alpha_1 a'_{i_1} + \ldots + \alpha_n a'_{i_n}. \tag{5}$$

A multiplication table for the a''s is given by the rule

$$a'_i a'_j = \beta_0 + \beta_1 a'_{i_1} + \ldots + \beta_m a'_{i_m},$$

whenever

$$a_i a_j = \beta_0 + \beta_1 a_{i_1} + \ldots + \beta_m a_{i_m}.$$

A new rule for multiplying any two real numbers is now determined, namely: Express each in the form (5), multiply the two polynomials, and reduce according to the new multiplication table.

* Hamel, *Mathematische Annalen*, Vol. LX, p. 459. Of course, this result is open to all the objections that can be made to the proposition that the continuum can be rearranged as a well-ordered set.

The number system determined in this fashion satisfies all the laws of operation of the ordinary real number system, for by letting each a' correspond to the a with the same subscript we set up a simple isomorphism between the two number systems. Moreover, the product of an irrational by a rational or of two rationals is the same in this system as in the ordinary one.

We may insure that the two number systems are distinct by selecting the a's in the first place so that $a_1 = \sqrt{2}$ and $a_2 = \sqrt{3}$, and then choosing the a''s so that $a'_1 = a_2$.

It is not difficult to set up a number system in which both multiplication and addition of irrationals are modified. It is not possible, however, to obtain one in which multiplication is the same as in the ordinary system while the rule for addition is altered for the irrational numbers. For by the distributive law,

$$a + b = a\left(1 + \frac{1}{a}b\right).$$

Since we are assuming that addition of rationals to irrationals is unaltered, $\left(1 + \frac{1}{a}b\right)$ is given by the usual rule. Since multiplication is unaltered, $a\left(1 + \frac{1}{a}b\right)$ is determined by the usual rule. Hence, $a + b$ is the same as in the ordinary system.

A Correction.

The number system which has been constructed in the article above may be used to point out an omission in the article on "A Set of Assumptions for Projective Geometry" by myself and Professor J. W. Young in Vol. XXX of this Journal. On page 364 we state an Assumption of Continuity as follows:

C. If there exists any non-modular net of rationality, at least one point Q of some line l and at least one net of rationality R on l containing Q is such that associated with every singly open cut $K(Q)$ in R is a point X_k such that: 1) X_k is on l; 2) if two cuts $K_1(Q)$ and $K_2(Q)$ are distinct, the points X_{k_1} and X_{k_2} are distinct; 3) if two cuts $K_1(Q)$ and $K_2(Q)$ are projective, the points X_{k_1} and X_{k_2} form a homologous pair.

This, together with the assumptions of alignment and extension and the assumption that there is not more than one chain on a line, we stated to be sufficient to determine the real projective geometry.

If we construct an analytic projective space based on the number system described above, we find, however, that it satisfies all of these assumptions. This is obvious with the exception, perhaps, of the third part of Assumption C. If the net R of that assumption be taken as R $(0\ 1\ \infty)$, any projectivity which transforms this net into itself may be written

$$x' = \frac{a x + b}{c x + d},$$

where a, b, c, d are rational. Such a transformation gives the same result in the number system based on the a's as in the ordinary one, and hence C, 3) is satisfied. Therefore, in order that our set of assumptions for projective geometry be categorical, Assumption C must be strengthened by a provision that order relations are preserved also by projective transformations among irrational points.

I have noted the following other errata in the paper on Projective Geometry:

Part of the proof of Theorem 14, page 355, is omitted; the part lacking is, however, well known.

On page 359, second line of Theorem 19, read "net" instead of "plane".

On the second line of Theorem 20 insert "of the net" after "(line)".